Journal of Engineering Physics and Thermophysics, Vol. 66, No. 6, 1994

FLUCTUATION MODEL OF A NONEQUILIBRIUM TWO-PHASE CHANNEL FLOW

F. A. Krivoshei

UDC 532.529:533.15

An ill-posed Cauchy problem for a model of a nonequilibrium two-phase flow in the barotropic approximation is transformed into a well-posed problem by changing the type of the initial hyperbolic equations. Approximation of fluctuations of the phase velocities by a random delta-correlated process and averaging of the equations over its realizations generate a system of parabolic equations. Results of numerical integration of this system are compared with experiment and calculations by well-known models.

1. The ill-posedness of the Cauchy problem due to loss of hyperbolicity of the system of equations for a two-phase flow in the approximation of phase barotropy was investigated in a number of works to eliminate or at least restrict its domain [1–4]. Since the difficulties stem from the hypothesis of equality of phase pressures, it seems natural to employ a model with unequal pressures [4, 5]. However, in this case an additional closing relation is needed that would link the phase pressures. In particular, the approach of [5] requires knowledge of the radius of curvature of the interface in real two-phase flows. It is the absence of reliable information on this parameter that makes a model with unequal pressures practically useless for applications.

The Cauchy problem can be made well-posed by changing the type of the initial hyperbolic equations. In this case the issue of ill-posedness due to loss of hyperbolicity of the system is formally eliminated, since it is no longer hyperbolic *a priori* (before integration). The approach suggested is based on two procedures: introduction of fluctuating parameters of the medium into the equations of a stochastic approximation and subsequent averaging of the obtained stochastic hyperbolic equations over the realizations of the random parameters. This procedure results in a system of parabolic equations for the average values of the sought parameters of the flow, and it is necessary to find out whether this system has properties that prevent the development of instability of numerical solutions.

2. The initial system of hyperbolic equations is written in the approximation of phase barotropy in matrix form:

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{1}{\rho_i} \frac{\partial p}{\partial x} = F_i , \tag{1}$$

$$\frac{\partial p}{\partial t} + R \sum_i \varphi_i \frac{\partial v_i}{\partial x} + R \sum_i \frac{\varphi_i v_i}{\rho_i a_i^2} \frac{\partial p}{\partial x} + R (v_2 - v_1) \frac{\partial \varphi_2}{\partial x} = F_3 , \tag{2}$$

$$\frac{\partial \varphi_2}{\partial t} + \varphi_2 \left(1 - \frac{R \varphi_2}{\rho_2 a_2^2} \right) \frac{\partial v_2}{\partial x} - \frac{R \varphi_2}{\rho_2 a_2^2} (1 - \varphi_2) \frac{\partial v_1}{\partial x} +$$

$$+ \frac{\varphi_2}{\rho_2 a_2^2} \left(v_2 - R \sum_i \frac{\varphi_i v_i}{\rho_i a_i^2} \right) \frac{\partial p}{\partial x} + \left[v_2 - \frac{\varphi_2 R (v_2 - v_1)}{\rho_2 a_2^2} \right] \frac{\partial \varphi_2}{\partial x} = F_4 , \tag{3}$$

Institute of Engineering Thermophysics, Academy of Sciences of Ukraine, Kiev, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 66, No. 6, pp. 651-656, June, 1994. Original article submitted October 28, 1992.

 1062-0125/94/6606-0578$12.50

$$\frac{\partial h_i}{\partial t} + \frac{R}{\rho_i} \sum_i \varphi_i \frac{\partial v_i}{\partial x} - \frac{1}{\rho_i} \left(v_i - R \sum_i \frac{\varphi_i v_i}{\rho_i a_i^2} \right) \frac{\partial p}{\partial x} +$$

$$+ \frac{R}{\rho_i} (v_2 - v_1) \frac{\partial \varphi_2}{\partial x} + v_i \frac{\partial h_i}{\partial x} = F_{i+4} , \tag{4}$$

where $i = 1, 2$ (1 refers to a liquid phase, 2 refers to a gas phase); F are the right-hand sides of the equations;

$$R = \left(\sum_i \frac{\varphi_i}{\rho_i a_i^2} \right)^{-1} , \quad \varphi_2 = 1 - \varphi_1 .$$

The velocities of the liquid and gas phases always fluctuate because of the statistical nature of processes in two-phase flows. The phase velocity v_i is expressed as a random function equal to the sum of the averaged and fluctuating terms:

$$v_i = \langle v_i \rangle + \delta v_i (t) .$$

Assuming that the phase velocities can be described by a delta-correlated Gaussian process, we introduce the correlation function

$$\langle \delta v_i (t_1) \, \delta v_i (t_2) \rangle = 2\sigma_i^2 \, \delta (t_1 - t_2) , \tag{5}$$

where σ_i^2 is the variance of the velocity v_i. Averaging the equations in system (1)–(4) over realizations of the random function v_i and evaluating the resultant statistical nonlinearities using the Furutsu–Novikov formula with correlation (5) taken into account, we obtain a system of parabolic equations for the average sought parameters v_i, ρ, φ_i, h_i:

$$\frac{\partial \langle v_i \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_i \rangle}{\partial x} - 2\sigma_i^2 \frac{\partial^2 \langle v_i \rangle}{\partial x^2} + \frac{1}{\rho_i} \frac{\partial \langle p \rangle}{\partial x} = \langle F_i \rangle , \tag{6}$$

$$\frac{\partial \langle p \rangle}{\partial t} + R \sum_i \langle \varphi_i \rangle \frac{\partial \langle v_i \rangle}{\partial x} + R \sum_i \frac{\langle \varphi_i \rangle \langle v_i \rangle}{\rho_i a_i^2} \frac{\partial \langle p \rangle}{\partial x} -$$

$$- 2R \sum_i \frac{\sigma_i^2 \langle \varphi_i \rangle}{\rho_i a_i^2} \left(\frac{\partial^2 \langle p \rangle}{\partial x^2} \right) + R (\langle v_2 \rangle - \langle v_1 \rangle) \frac{\partial \langle \varphi_2 \rangle}{\partial x} -$$

$$- 2R \left[(\sigma_2^2 - \sigma_1^2) \frac{\partial^2 \langle \varphi_2 \rangle}{\partial x^2} - \sum_i \frac{\sigma_i^2}{\varphi_i} \left(\frac{\partial \langle \varphi_i \rangle}{\partial x} \right)^2 \right] = \langle F_3 \rangle , \tag{7}$$

$$\frac{\partial \langle \varphi_2 \rangle}{\partial t} + \langle \varphi_2 \rangle \left[1 - \frac{R \langle \varphi_2 \rangle}{\rho_2 a_2^2} \right] \frac{\partial \langle v_2 \rangle}{\partial x} -$$

$$- \frac{R \langle \varphi_2 \rangle (1 - \langle \varphi_2 \rangle)}{\rho_2 a_2^2} \frac{\partial \langle v_1 \rangle}{\partial x} + \frac{\langle v_2 \rangle - \langle v_1 \rangle}{\sum_i \frac{\rho_i a_i^2}{\langle \varphi_i \rangle}} \frac{\partial \langle p \rangle}{\partial x} -$$

$$
-\frac{\sigma_2^2-\sigma_1^2}{\sum_i \frac{\rho_i a_i^2}{\langle \varphi_i \rangle}} \frac{\partial^2 \langle p \rangle}{\partial x^2} + \left[\langle v_2 \rangle - \frac{R \langle \varphi_2 \rangle}{\rho_2 a_2^2} (\langle v_2 \rangle - \langle v_1 \rangle) \right] \times
$$

$$
\times \frac{\partial \langle \varphi_2 \rangle}{\partial x} - 2 \left[\sigma_2^2 - \frac{R \langle \varphi_2 \rangle}{\rho_2 a_2^2} (\sigma_2^2 - \sigma_1^2) \right] \frac{\partial^2 \langle \varphi_2 \rangle}{\partial x^2} +
$$

$$
+ 2 \sum_i \frac{\rho_i a_i^2 \sigma_i^2}{\langle \varphi_i \rangle^2} \left(\sum_i \frac{\rho_i a_i^2}{\langle \varphi_i \rangle} \right)^{-1} \left(\frac{\partial \langle \varphi_2 \rangle}{\partial x} \right)^2 = \langle F_4 \rangle , \tag{8}
$$

$$
\frac{\partial h_i}{\partial t} + \frac{R}{\rho_i} \sum_i \langle \varphi_1 \rangle \frac{\partial \langle v_i \rangle}{\partial x} - \frac{1}{\rho_i} \left(\langle v_i \rangle - R \sum_i \frac{\langle \varphi_i \rangle \langle v_i \rangle}{\rho_i a_i^2} \right) \times
$$

$$
\frac{\partial \langle p \rangle}{\partial x} + \frac{R}{\rho_i} (\langle v_2 \rangle - \langle v_1 \rangle) \frac{\partial \langle \varphi_2 \rangle}{\partial x} + \langle v_i \rangle \frac{\partial \langle h_i \rangle}{\partial x} + \frac{2}{\rho_i} \times
$$

$$
\times \left(\sigma_i^2 - R \sum_i \frac{\langle \varphi_i \rangle \sigma_i^2}{\rho_i a_i^2} \right) \frac{\partial^2 \langle p \rangle}{\partial x^2} - \frac{2R}{\rho_i} \times
$$

$$
\times \left[(\sigma_2^2 - \sigma_1^2) \frac{\partial^2 \langle \varphi_2 \rangle}{\partial x^2} - \sum_i \frac{\sigma_i^2}{\langle \varphi_i \rangle} \left(\frac{\partial \langle \varphi_2 \rangle}{\partial x} \right)^2 \right] - 2 \sigma_i^2 \frac{\partial^2 \langle h_i \rangle}{\partial x^2} = \langle F_{i+4} \rangle . \tag{9}
$$

Thus, averaging of Eqs. (1) – (4) over the realizations of the stochastic parameter v_i generates a parabolic system. In particular, averaging the total derivative of the velocity generates the Bürgers–Hopf operator for the average velocity, so that Eqs. (6) are similar to a Bürgers–Hopf system. The appearance of coefficients σ_i^2 of the second derivatives having dimensions of kinematic viscosity is associated with the "viscosity" technique in gasdynamics [6]. It is interesting to note that the phase velocities (coefficients of the first spatial derivatives of the pressure) are replaced by the variances σ_i^2 at the second derivatives. Similarly, the slip $\langle v_2 \rangle - \langle v_i \rangle$ is replaced by the slip of the corresponding variances $\sigma_2^2 - \sigma_1^2$.

Investigation of the conditions for well-posedness and the properties of solutions of boundary-value problems for nonlinear parabolic systems of type (6) – (9) involves substantial difficulties, and even simpler systems of two equations of Bürgers–Hopf type with viscosity [6] have been studied inadequately. To test well-posedness of the present model, results of numerical integration are compared with experiment and results from well-known models. Here, system (6) – (9) is supplemented by initial and boundary conditions and closing relations that enter the right-hand sides of the equations and are determined by the formulation of the specific problem.

3. As an example of this problem, the well-known problem of the outflow of a boiling liquid [7, 9] will be formulated for system (6) – (9). A tube of length L and constant cross-sectional area is closed at both ends by membranes and filled with homogeneous water at the pressure p_0 and a temperature T_{10} lower than the saturation temperature T_s. At $t = 0$ the membrane at one of the ends is destroyed, and at $t > 0$ the boiling water flows out into surroundings at the pressure p_∞ ($p_\infty \ll p_0$). The flow is assumed to be adiabatic, and the friction forces on the tube walls are neglected.

The initial conditions for the homogeneous liquid have the form

$$
t = 0 : p(x, 0) = p_0 , \quad T(x, 0) = T_{10} , \quad \varphi_1(x, 0) = 1 , \quad \varphi_2(x, 0) = 0 . \tag{10}
$$

The boundary condition at the closed end of the tube is the condition of the absence of any leaks

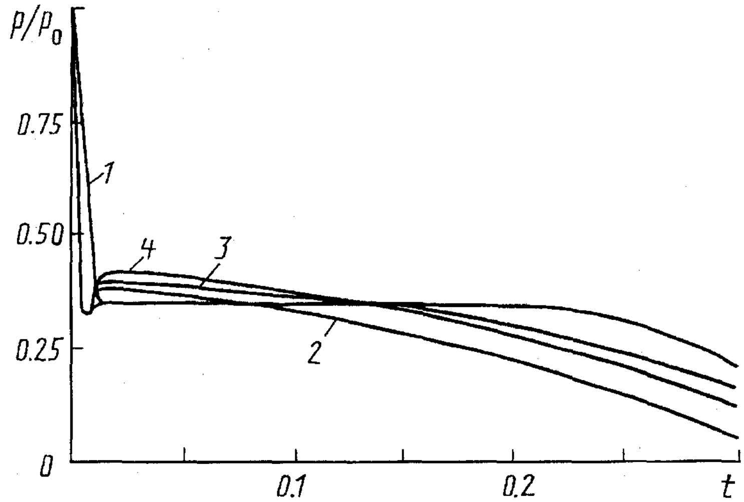

Fig. 1. Experimental and calculated pressure curves in a tube with seal failure: 1) experiment [11]; 2) result of [7] at $p_0 = 6.9$ MPa, $T = 515$ K; 3) present result at $p_0 = 6.9$ MPa, $T = 515$ K, $\sigma_1^2 = 20$, $\sigma_2^2 = 25$; 4) the same, at $\sigma_1^2 = 60$, $\sigma_1^2 = 75$. t, sec.

$$x = 0: v = 0. \tag{11}$$

The boundary condition at the open end of the tube is equality of the pressures at the exit cross section of the tube and in the ambient medium

$$x = L: p = p_\infty. \tag{12}$$

Solution of system (6)–(9) with boundary conditions (10)–(12) and closing relations from [2, 7, 9] is obtained by a Lax–Wendroff difference scheme with artificial viscosity that includes the effects of nonlinearities [10]. The stability condition for this scheme

$$\frac{(|v| + \bar{a})\Delta t}{\Delta x} < \left(1 + \frac{b^2}{4}\right)^{1/2} - \frac{b}{2} \tag{13}$$

is more limiting than the Courant condition. Here Δx and Δt are the space and time steps, respectively. Since numerical solutions depend on the parameters σ_i^2, it is natural to relate them to stability condition (13) and the necessary condition of approximation for a parabolic system [6]

$$(\Delta x)^2/2\Delta t = \sigma_i^2. \tag{14}$$

The indeterminacy in expression (14) can be eliminated by virtue of the inequality $\sigma_2^2 > \sigma_1^2$. Then, with a fixed Δx, the value of σ_2^2 in the right-hand side of (14) corresponds to a smaller time step, which is important in the initial stages of boiling with a highly nonequilibrium flow. Assuming $b = \sigma_2^2/\sigma_1^2$ ($\sigma_1 \neq 0$), we obtain necessary conditions for choosing Δx and Δt with account for the parameters σ_i^2.

Figure 1 shows experimental and calculated pressure curves in a fixed cross section ($x = 1.39$ m from the closed end) in the case of seal failure of a tube of length $L = 4.1$ m filled with water at $p_0 = 6.9$ MPa. It is seen that in accordance with the prediction made in [8] on the effect of relative motion of the phases, inclusion of this factor gives better agreement of the calculated and experimental data than the model suggested in [7]. The values $\sigma_1^2 = 20$, $\sigma_2^2 = 25$ are close to optimum in the sense of the stability of the scheme and the amount of calculations. A decrease in σ_i^2 results in disruption of the stability of the scheme in the time step and degeneration of the parabolic system. A increase in σ_i^2 leads to an unnecessary decrease in the time step and increase in the amount of calculations.

4. To conclude the present work, we consider the possibility of reducing a parabolic system of type (6) – (9) to a system of the ordinary differential equations. This approach was used, in particular, to analyze solutions of the Hopf [6] and Oberbeck–Boussinesq [12] equations.

The following dimensionless variable will be introduced:

$$\xi_i = \frac{x}{L_i} - \frac{\sigma_i^2 t}{L_i^2},$$

where L_i is the scale of velocity fluctuations in the i-th phase; $\tau_i = L_i^2/\sigma_i^2$ is the characteristic correlation time. Reformulation of system (6) – (9) in the variable ξ_i gives a system of ordinary differential equations in the two variables ξ_1 and ξ_2. It will be shown now that under certain conditions it is possible to pass to one variable, having found a relation between ξ_1 and ξ_2. We consider the relation

$$\frac{\xi_1}{\xi_2} = \frac{L_1}{L_2}\left(x - \frac{\sigma_1^2 t}{L_1}\right)\left(x - \frac{\sigma_2^2 t}{L_2}\right)^{-1}$$

and its transformation

$$\frac{\xi_1}{\xi_2} = \frac{L_2}{L_1}\left[1 - \left(\frac{\sigma_1^2 t}{L_1} - \frac{\sigma_2^2 t}{L_2}\right)\left(x - \frac{\sigma_2^2 t}{L_2}\right)^{-1}\right]. \tag{15}$$

According to [13], the following inequality is valid:

$$\frac{\sigma_1^2}{L_1} << \frac{\sigma_2^2}{L_2}, \tag{16}$$

which means in this case that the velocity of large-scale fluctuations of the carrier phase is much smaller than that of small-scale fluctuations of the disperse phase. Then, by virtue of inequality (16), expression (15) can be written as

$$\frac{\xi_1}{\xi_2} \approx \frac{L_2}{L_1}\left[1 + \frac{\sigma_2^2 t}{L_2}\left(x - \frac{\sigma_2^2 t}{L_2}\right)^{-1}\right].$$

In some practical cases, we are interested in the values of the sought flow parameters near the exit at large $x \sim l$, where l is the channel length. Then the inequality

$$x >> \frac{\sigma_i^2 t}{L_i}, \tag{17}$$

is valid. It is in complete agreement with a similar inequality used in [13] for turbulent pulsations. Inequality (17) means that in relation to local properties of the flow (i.e., fluctuations), the main flow of the carrier phase with the scale $x \sim l$ can be assumed to be steady. This approximation also agrees with the meaning of the delta-correlation of fluctuations of the phase velocity v_i caused by the smallness of the change in the functionals of the process in a time of the order of its characteristic correlation time τ_i. Then with inequality (17) taken into account, we have

$$\frac{\sigma_2^2 t}{L_2}\left(x - \frac{\sigma_2^2 t}{L_2}\right)^{-1} << 1, \quad \frac{\xi_1}{\xi_2} \approx \frac{L_2}{L_1}. \tag{18}$$

Introducing relation (18) into the equations with the two variables ξ_1 and ξ_2, we obtain a system of ordinary differential equations in one variable with the scale factor L_2/L_1 as a coefficient.

NOTATION

a, velocity of propagation of acoustic perturbations; $\overline{a}$, "frozen" velocity of sound; b, dimensionless constant; h, enthalpy; L, tube length, scale of velocity fluctuations; p, pressure; R, complex; T, temperature; t, time; v, velocity; x, coordinate; Δ, step of the difference scheme; δ, fluctuating component of the velocity, delta-function; ξ, dimensionless variable; ρ, phase density; σ^2, variance of the velocity; τ, characteristic correlation time; φ, volume phase concentration.

REFERENCES

1. R. I. Nigmatulin, Dynamics of Multiphase Media 1 [in Russian], Part 1, Moscow (1987).
2. S. Banerjee and W. T. Hancox, in: Heat Transfer-1978, Proc. 6th Int. Heat Transfer Conf., Toronto, Canada (1978), Vol. 6, pp. 313-337.
3. L. A. Klebanov, A. E. Kroshilin, B. I. Nigmatulin, and R. I. Nigmatulin, Prikl. Mat. Mekh., **46**, 83-96 (1982).
4. V. H. Ransom and D. L. Hicks, J. Comput. Phys., **53**, 124 (1984).
5. J. D. Ramshaw and J. A. Trapp, Nucl. Sci. Eng., **66**, 93-102 (1978).
6. B. L. Rozhdestvenskii and N. N. Yanenko, Systems of Quasilinear Equations and Their Application to Gas Dynamics [in Russian], Moscow (1978).
7. B. I. Nigmatulin and K. I. Soplenkov, Teplofiz. Vysok. Temp., **18**, No. 1, 118-131 (1984).
8. R. I. Nigmatulin, Dynamics of Multiphase Media, [in Russian], Part 2, Moscow (1987).
9. A. E. Kroshilin, V. E. Kroshilin, and B. I. Nigmatulin, Teplofiz. Vysok. Temp., **22**, No. 2, 355 (1984).
10. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems [Russian translation], Moscow (1972), pp. 93-102.
11. A. R. Edwards and O'Brien, J. Brit. Nucl. Soc., **9**, 125-135 (1970).
12. D. D. Joseph, Stability of Fluid Motions [Russian translation], Moscow (1976).
13. L. D. Landau and E. M. Lifshits, Hydrodynamics [in Russian], Moscow (1988).